Monetary Policy and the Transition to Rational Expectations

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(Comments are welcome)

Abstract

Under the assumption of bounded rationality, economic agents learn from their mistaken predictions of the past by combining new and old information in the formation of new beliefs. The purpose of this paper is to examine how the policy maker, by affecting the private agents learning process, determines the speed at which the economy converges to the rational expectation equilibrium. Our findings show the relevance of this transition period when we look at a criterion for evaluating monetary policy decisions.

1 Introduction

The fact that monetary policy decisions could affect the real economy in the short run is widely accepted. Part of the literature has obtained this result explicitly incorporating frictions, such as nominal price rigidities, in a dynamic general equilibrium framework under the rational expectations (RE) hypothesis.

Recently, this problem has been analyzed questioning the assumption that the public can make unbiased predictions of the future course of the economy. Such predictions, has been said, would be possible if people had observed the reactions of the policy makers to various economic conditions over a long period of time. However, this would not always be the case. It can be argued that, for example, in the presence of regime policy shifts the public needs to learn

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about the new regime: in the early stages of this learning process, previously held public beliefs could lead to biased predictions. Evans and Honkapohja (2002) and Bullard and Mitra (2001)[1], have shown in presence of boundedly rational private agents, under which hypothesis the policy maker could induce an improvement of agents' forecasts over time and determines a convergence of peoples' guess to the RE equilibrium (*E-Stability*). These papers have studied the asymptotic properties of the equilibria obtained under learning but no results are derived about the dynamics along the convergence process, the transition to the REE.

The purpose of this work is to examine the role of the policy maker in determining the speed of convergence of the private agents learning process and the implications in terms of the monetary policy design problem. I show that policies that asymptotically determine the same REE could in the transition generate extremely different dynamics in the real economy. In particular, the central result of this paper is that, if policy decisions depend on social welfare and the policy maker ignores the private agents' learning dynamics in the transition, considerable welfare losses could be incurred. Intuitively, the fact that under a certain policy the private agents' learning process (and, in general, the economy) converges after 100.000 years, while under another one the transition lasts only 1 year, is relevant in terms of policy decisions.

I study the transition in the learning process to the REE by adapting arguments described by Marcet and Sargent (1992), which in turn are based on theoretical results of Benveniste, Metivier and Priouret (1990). The underlying mechanism is the following: the policy maker chooses a policy rule; this policy determines the eigenvalues of the associated ordinary differential equation (ODE) at the fixed point; the eigenvalues incorporate information about the speed at which agents learn.

The paper is organized as follows. Section 2 presents the general monetary policy design problem under discretion and rational expectation hypothesis. Section 3 presents the model with boundedly rational private agents and shows that, if the policy maker does not take into account his role in determining the speed of convergence of the learning process to the REE, private agents could need many periods in order to learn the REE. In section 4 I show how the policy maker should take into account the fact that his policy decisions will affect the transitions dynamics. In section 5, I consider a policy that allows the policy maker to increase (or reduce) the speed of convergence without affecting the long run equilibrium (i.e., the REE equilibrium at which the learning process converges) and analyze the implications in terms of policy responses to changes in expected inflation and expected output gap, and in terms of cost-push shocks and government expenditure shocks.

[1] An earlier paper by Howitt (1992) has already showed that under some interest rate rules the rational expectation equilibrium is not learnable.

2 The Monetary Policy Design Problem

Choosing between policies based on simple rules or derived as a solution of a specified optimization problem is the point of departure for the analysis of how to conduct monetary policy. We will start by considering the solution of the optimal monetary policy problem without commitment (discretionary policies), where any promises made in the past by the policy maker do not constrain current policies[2].

2.1 A Baseline Model

Much of the recent theoretical analysis of monetary policy has been conducted using the "New Phillips curve" paradigm, reviewed in Clarida, Galí and Gertler (1999) and Woodford (1999). The baseline framework is a dynamic general equilibrium model with money and temporary nominal price rigidities. We will consider the linearized reduced form of the economy with competitive monopolistic firms, staggered prices and private agents that maximize intertemporal utilities. From the private agents side we have an intertemporal "IS curve" and an Aggregate Supply (AS) modeled by an expectations-augmented "Phillips Curve":

$$\text{IS:} \quad x_t = \widehat{E}_t x_{t+1} - \varphi\left(i_t - \widehat{E}_t \pi_{t+1}\right) + g_t \tag{2.1}$$

$$\text{AS:} \quad \pi_t = \alpha x_t + \beta \widehat{E}_t \pi_{t+1} + u_t \tag{2.2}$$

where x_t is the output gap, that is the log deviation of actual output (y_t) from the potential output (z_t) (i.e., the level of output that would arise if wages and prices were perfectly competitive and flexible)

$$x_t = y_t - z_t \tag{2.3}$$

π_t is actual inflation at time t, $\widehat{E}_t \pi_{t+1}$ is the level of inflation that private agents expects for period $t+1$, given the information at time t. Similarly $\widehat{E}_t x_{t+1}$ is the level of output gap that private agents expect for period $t+1$, given the information at time t; i_t is the short term nominal interest rate and is taken to be the instrument for monetary policy; u_t is a cost-push shock and g_t is a demand shock, with

$$u_t = \rho_u u_{t-1} + \varepsilon_t \tag{2.4}$$

$$g_t = \rho_g g_{t-1} + \upsilon_t \tag{2.5}$$

$$\upsilon_t \sim N(0, \sigma_\upsilon) \quad \text{and i.i.d} \tag{2.6}$$

[2] I leave for future research a general study of the transition of learning process for monetary policy problem under commitment

$$\varepsilon_t \sim N(0, \sigma_\varepsilon) \quad \text{and i.i.d} \tag{2.7}$$

The IS relation approximates the Euler equation characterizing optimal aggregate consumption choices and the parameter φ can be interpreted as the rate of intertemporal substitution. The AS relation approximate the aggregate pricing equation emerging from monopolistically competitive firms' optimal behavior in Calvo's model of staggered price determination[3].

The central bank objective function depends on the squared deviations of output gap and inflation from their respective targets[4].

$$\underset{x_{t+i},\pi_{t+i}}{Max} - E_t \sum_{i=0}^{\infty} \beta^i \frac{1}{2} \left[\pi_{t+i}^2 + \lambda (x_{t+i} - \overline{x})^2 \right] \tag{2.8}$$

where $\overline{x}$ allows for a possible deviation of social optimal from potential output, and the target inflation is zero.

The policy problem consists in choosing the time path for the instrument i_t to engineer time paths of target variables π_{t+i} and $(x_{t+i} - \overline{x})$ that maximizes the objective function subject to the constraints 2.1 and 2.2.

In this context, a central bank operating under discretion chooses the current interest rate by reoptimizing every period. Assuming that the policy maker cannot credibly manipulate beliefs, he would maximize his objective function taking private sector expectations as given. The solution of this problem yields the following optimality conditions[5]

$$\pi_t = \frac{\lambda\alpha}{(\lambda+\alpha^2)}\overline{x} + \frac{\lambda\beta}{(\lambda+\alpha^2)}\widehat{E}_t\pi_{t+1} + \frac{\lambda}{(\lambda+\alpha^2)}u_t \tag{2.9}$$

$$x_t = \frac{\lambda}{(\lambda+\alpha^2)}\overline{x} - \frac{\alpha\beta}{(\lambda+\alpha^2)}\widehat{E}_t\pi_{t+1} - \frac{\alpha}{(\lambda+\alpha^2)}u_t \tag{2.10}$$

The optimal outcome could be written as a feedback policy that relates the policy instrument i_t to the current state of the economy and the expectations of private agents:

$$i_t = \gamma^\star + \gamma_x^\star \widehat{E}_t x_{t+1} + \gamma_\pi^\star \widehat{E}_t \pi_{t+1} + \gamma_u^\star u_t + \gamma_g^\star g_t \tag{2.11}$$

[3] Inflation is increasing with the output gap as price are set as a markup over real marginal costs, which are increasing with the output gap. Higher expected inflation raises current inflation, as price setters cannot fully adjust to current shocks.

[4] I will consider λ as an exogenous policy parameter, as is often done in the monetary policy literature. An alternative approach consists in obtaining it as the result of the general equilibrium problem where the value of λ would depends on the representative consumer preferences and firms price setting rules.

[5] To obtain this result note that, after substituting the constraints 2.1 and 2.2 in to the loss function, the problem becomes

$\underset{i_t}{Max} - \frac{1}{2}\left[(\alpha E_t x_{t+1} - \varphi i_t + g_t + (\beta+\varphi) E_t\pi_{t+1} + u_t)^2 + \lambda (E_t x_{t+1} - \varphi (i_t - E_t\pi_{t+1}) + g_t - \overline{x})^2 \right]$

s.t $E_t x_{t+1}, E_t \pi_{t+1}$ given

and the FOC is:

$i_t = -\frac{\lambda}{(\lambda+\alpha^2)\varphi}\overline{x} + \frac{1}{\varphi}E_t x_{t+1} + \left(1 + \frac{\alpha\beta}{(\lambda+\alpha^2)\varphi}\right) E_t\pi_{t+1} + \frac{g_t}{\varphi} + \frac{\alpha}{(\lambda+\alpha^2)\varphi}u_t$

and (2.9), (2.10) are obtained by substituting this expression into (2.1) and (2.2).

where

$$\begin{aligned}
\gamma^{\star} &= -\frac{\lambda}{(\lambda+\alpha^{2})\,\varphi}\overline{x} \\
\gamma_{x}^{\star} &= \frac{1}{\varphi} \\
\gamma_{\pi}^{\star} &= 1+\frac{\alpha\beta}{(\lambda+\alpha^{2})\,\varphi} \\
\gamma_{u}^{\star} &= \frac{\alpha}{(\lambda+\alpha^{2})\,\varphi} \\
\gamma_{g}^{\star} &= \frac{1}{\varphi}
\end{aligned} \tag{2.12}$$

The interest rate rule states that the policy maker should respond to expected inflation and output gap and observable exogenous shocks according to the optimal policy coefficients $\gamma^{\star}, \gamma_{x}^{\star}, \gamma_{\pi}^{\star}, \gamma_{u}^{\star}, \gamma_{g}^{\star}$, for this reason (2.11) is also called the *Optimal Expectations-based reaction function (*Evans and Honkapohja, 2002).

2.2 The Rational Expectation Equilibrium

Under Rational Expectation, the inflation and output gap equilibrium values, derived from (2.9) and (2.10), are the following:

$$\pi_t = a_\pi + b_\pi u_t \tag{2.13}$$

$$x_t = a_x + b_x u_t \tag{2.14}$$

where a_π, b_π, a_x, b_x are

$$\begin{aligned}
a_\pi &= \frac{\lambda\alpha}{(\lambda+\alpha^{2})-\lambda\beta}\overline{x} \\
b_\pi &= \frac{\lambda}{(\lambda+\alpha^{2})-\lambda\beta\rho_u} \\
a_x &= \frac{\lambda\,(1-\beta)}{(\lambda+\alpha^{2})-\lambda\beta}\overline{x} \\
b_x &= -\frac{\alpha}{(\lambda+\alpha^{2})-\lambda\beta\rho_u}
\end{aligned} \tag{2.15}$$

and the expected inflation and output gap would be

$$\widehat{E}_t\pi_{t+1} = E_t\pi_{t+1} = a_\pi + b_\pi\rho_u u_t \tag{2.16}$$

$$\widehat{E}_t x_{t+1} = E_t x_{t+1} = a_x + b_x\rho_u u_t \tag{2.17}$$

If we substitute the value of the conditional expectations (2.16) and (2.17) into (2.11), the optimal policy rule could be written as:

$$i_t = \gamma^R + \gamma_u^R u_t + \gamma_g^R g_t \tag{2.18}$$

$$
\begin{aligned}
\gamma^R &= \frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} \\
\gamma_u^R &= \frac{\varphi\lambda\rho_u+(1-\rho_u)\,\alpha}{((\lambda+\alpha^2)-\lambda\beta\rho_u)\,\varphi} \\
\gamma_g^R &= \frac{1}{\varphi}
\end{aligned}
\tag{2.19}
$$

Equation (2.18) says that the policy maker should offset demand shocks (g_t) adjusting the nominal interest rate in order to neutralize any shock to the IS curve; for what concerns the supply shocks (u_t), we have a trade-off between inflation and output gap variability. Since this optimal policy rule involves only the fundamentals of the economy (demand and supply shocks), (2.18) could be defined as the *Optimal fundamentals-based reaction function* under rational expectations (Evans and Honkapohja (2002))[6].

2.3 RE and the Expectations-based reaction function

We would like to stress the fact that under rational expectations, we have a problem of indeterminancy of *optimal expectations-based policy rules*.

Simply consider a generic *expectations-based* policy rule of the form:

$$
i_t = \gamma + \gamma_x \widehat{E}_t x_{t+1} + \gamma_\pi \widehat{E}_t \pi_{t+1} + \gamma_u u_t + \gamma_g g_t \tag{2.20}
$$

Now, assuming rational expectations, we could substitute (2.16) and (2.17) in to (2.20) obtaining the following policy rule:

$$
i_t = (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) + (\gamma_x b_x \rho_u + \gamma_\pi b_\pi \rho_u + \gamma_u)\, u_t + \gamma_g g_t \tag{2.21}
$$

Equation (2.21) should be consistent with the *optimal fundamentals-based policy rule* derived in equation (2.18). That is

$$
\begin{aligned}
\gamma^R &= (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) \\
\gamma_u^R &= (\gamma_x b_x \rho_u + \gamma_\pi b_\pi \rho_u + \gamma_u) \\
\gamma_g^R &= \gamma_g
\end{aligned}
\tag{2.22}
$$

Notice that (2.22) is a system of three equations on five unknowns. Obviously this system has multiple solutions, meaning that there is a continuum of *expectations-based policy rules* of the form (2.20) that are consistent with the optimal discretionary rational expectation equilibrium.

[6] Many autors (see for example Woodford (1999)) have shown that this interest rate rule leads to indeterminacy, i.e. a multiplicity of rational expectation equilibria. This means, under this policy rule, there exists other stationary REE.

3 Learning Models and Policy Analysis

The current standard hypothesis about expectations in the monetary policy design problem, as we have seen, is the Rational Expectation hypothesis, meaning that agents do not make systematic forecasting errors and their guesses about the future are on average correct.

In this paper I focus on a different approach to model expectations; I assume, while the policy maker is perfectly rational, private agents do not initially have rational expectations, but instead they form forecasts by using recursive learning algorithm and these forecast functions are revised over time as new data become available. Under this approach, the rational expectations equilibrium becomes, possibly, a limit of the temporary learning equilibrium. In contrast, rational expectation approach retains rational expectation equilibrium continuously over time.

3.1 Bounded Rationality and Rational Expectations

In the literature, economic models with adaptive learning agents have been used with two different purposes. First, by providing an asymptotic justification for the RE hypothesis and a selection device in presence of multiple REE, they have been used to offer a rationale for rational expectations; second, to offer a description of the behaviour of the economy not only asymptotically, but even during the transition to the REE, showing dynamics that are not available under perfect rationality and that could be of empirical relevance.

The first approach is, for example, in Evans and Honkapohja (2002), Bullard and Mitra (2001) and Bullard and Mitra (2002), the second one is used by Marcet and Sargent (1989b), Marimon (1997), Sargent (1999) and Marcet and Nicolini (2001).

In this paper I will show that considering learning in a model of monetary policy design is particularly important in order to describe not only the rational expectation equilibrium to which we could converge under "plausible" learning schemes, but even to describe the dynamics in the transition to this equilibrium.

3.2 The Learning Mechanism

The methodology I will follow to model the learning process is the one considered by Marcet and Sargent (1989a) and Evans and Honkapohja (2001, 2002). Private agents, using data generated by the system in which they operate, update their forecasts through recursive least squares alghorithms. This procedure is an example of adaptive real-time learning, which basic idea is that agents follow a standard statistical or econometric procedure for estimating the perceived law of motion of the main economic variables. The forecasts needed in decision-making are then computed from the estimated law of motion. The learning mechanism says how new information is incorporated into the statistics.

Let's go back now to the monetary policy design problem under discretion. A first easy way to introduce boundedly rational private agents in the context we have analyzed before is to assume that "the policy maker does not make active use of learning behavior of the agents" (Evans and Honkapohja (2002)). In this case, the only difference with the traditional monetary policy problem under discretion and rational expectation hypothesis concerns the private agents' forecasting functions.

Let's define

$$Y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} \tag{3.1}$$

than we could rewrite (2.9) and (2.10) in a compact way as

$$Y_t = g\left(\widehat{E}_t Y_{t+1}, u_t, g_t, \eta\right) \tag{3.2}$$

where η is a vector of parameters in the economy that includes parameters of monetary policy.

Under rational expectations, if we define

$$\Phi_\pi = \begin{pmatrix} a_\pi \\ b_\pi \end{pmatrix} \tag{3.3}$$

$$\Phi_x = \begin{pmatrix} a_x \\ b_x \end{pmatrix} \tag{3.4}$$

we could write (2.13) and (2.14) in the following way:

$$Y_t = g\left(u_t, \Phi_\pi, \Phi_x\right) = \begin{pmatrix} \Phi_\pi' \\ \Phi_x' \end{pmatrix} \begin{pmatrix} 1 \\ u_t \end{pmatrix} \tag{3.5}$$

Under least squares learning hypothesis, we assume that the private agents do not know the effective value of the a_π, b_π, a_x, b_x coefficients, but estimate them through recursive least square regressions. In this case, agents expectations are given by:

$$\widehat{E}_t Y_{t+1} = h\left(u_t, \Phi_{\pi,t}\left(\mu\right), \Phi_{x,t}\left(\mu\right)\right) \tag{3.6}$$

where $\Phi_{\pi,t}(\mu)$ and $\Phi_{x,t}(\mu)$ are certain statistics inferred from past data and h is the forecast function that depends on today's state and the statistics. These statistics are generated by learning mechanisms f_x and f_π

$$\Phi_{\pi,t}\left(\mu\right) = f_\pi\left(\Phi_{\pi,t}\left(\mu\right), u_t, \mu\right) \tag{3.7}$$

$$\Phi_{x,t}\left(\mu\right) = f_x\left(\Phi_{x,t}\left(\mu\right), u_t, \mu\right) \tag{3.8}$$

where μ are certain learning parameters that govern how past data is used into forming the statistics. Equations (3.2), (3.6), (3.7) and (3.8) determine the equilibrium sequence for given learning parameters μ.

Three points should be enfatized:

1. The statistics are only functions of observed data, not of the true model or the true parameters Φ_x, Φ_π.
2. The learning mechanisms f_x and f_π says how new information is incorporated into the statistics
3. The learning parameters μ govern for example the weight that is given to recent information.

In the context of our model, if we define

$$Z_t = \begin{pmatrix} 1 \\ \rho_u u_t \end{pmatrix} \tag{3.9}$$

the function h will be

$$\widehat{E}_t Y_{t+1} = \begin{pmatrix} \Phi'_{\pi,t} \\ \Phi'_{x,t} \end{pmatrix} \times Z_t \tag{3.10}$$

I assume learning mechanisms f_x and f_π

$$\Phi_{\pi,t} = \Phi_{\pi,t-1} + t^{-1} R_{t-1}^{-1} Z_{t-2} \left(\pi_{t-1} - Z'_{t-2} \Phi_{\pi,t-1} \right) \tag{3.11}$$

$$\Phi_{x,t} = \Phi_{x,t-1} + t^{-1} R_{t-1}^{-1} Z_{t-2} \left(x_{t-1} - Z'_{t-2} \Phi_{x,t-1} \right) \tag{3.12}$$

where

$$R_t = R_{t-1} + t^{-1} \left(Z_{t-1} Z'_{t-1} - R_{t-1} \right) \tag{3.13}$$

That is, the perceived law of motion (PLM) of inflation and output gap are updated by a term that depends on the last prediction errors[7] weigthed by the *gain sequence* t^{-1}. It is well known that in this case the adaptive procedure is the result of a least squares regression of inflation and output gap on a constant and the cost push shocks.

By substituting the actual law of motion (ALM) of inflation and output gap, as described by (2.1) and (2.2), into equations (3.11) and (3.12) it is easy to see that the estimates of the learning parameters at time t depend on the past values of the monetary policy instrument i_t.

[7] This formula implies that private agents do not use today's inflation and output gap in order to formulate their forecasts. Moreover we are assuming that the agents know the actual law of motion of the exogenous shocks u_t. This means that we assume, for simplicity, that private agents do not learn about the behavior of the cost push shocks. If not, it too can be estimated by a separate regression of u_t on u_{t-1}. These assumptions are made purely for convenience, and they are often made in models of learning, since they simplify solving the model. The dynamics of the model are unlikely to change.

3.3 E-stability of the REE

An important aspect of recursive learning is that, under some conditions, the learning mechanism converges to rational expectations and the learning equilibrium converges to the REE (**E-stability**). In order to determine the learnability of a rational expectation equilibrium I will follow the literature on least-squares learning (Marcet and Sargent (1989) and Evans and Honkapohja (2001)). The basic concept is the E-Stability result, that states that if the equilibrium is expectational stable (E-Stable), than the recursive least squares learning equilibrium is locally convergent to the rational expectations equilibrium. Stability under learning, or learnability of the REE, is desirable because it indicates that if agents are learning from past data, their forecasts will converge over time to the REE.

Evans and Honkapohja (2002) show that, in an economy where private agents are learning in the way we have just described, if the policy maker assumes (incorrectly) that private agents are fully rational (i.e. the policy maker follows the *optimal fundamentals-based reaction function* (2.18)), then the rational expectations equilibrium is not only undetermined but even unstable under learning dynamics. In other words, in such an economy, small expectational errors by private agents become magnified by the policy, and the probability to converge to rational expectation is equal to zero. If, instead, the policy maker follows the specific *expectations-based policy rule (2.11),* the private agents' learning mechanism converges to the rational expectations and the resulting equilibrium is E-stable. More in general, Bullard and Mitra (2002) have shown under which conditions, a generic expectations-based monetary policy rule[8] determines learnability.

Let's consider the set of expectations-based policy rules characterized by the following structure:

$$i_t = \gamma + \gamma_x \widehat{E}_t x_{t+1} + \gamma_\pi \widehat{E}_t \pi_{t+1} + \gamma_u u_t + \gamma_g g_t \tag{3.14}$$

and let's rewrite the actual law of motion (ALM) of inflation and output gap, obtained by substituting (3.14) into (2.1) and (2.2), in a compact way:

$$Y_t = Q + R \times \widehat{E}_t Y_{t+1} + S u_t \tag{3.15}$$

where Q and S are vectors and R is a matrix that depends on the policy parameters $\gamma, \gamma_x, \gamma_\pi, \gamma_u, \gamma_g$

$$Q = \begin{bmatrix} -\alpha\varphi\gamma \\ -\varphi\gamma \end{bmatrix}, \quad S = \begin{bmatrix} (1-\alpha\varphi\gamma_u) \\ -\varphi\gamma_u \end{bmatrix} \tag{3.16}$$

$$R = \begin{bmatrix} (\beta + \alpha\varphi(1-\gamma_\pi)) & \alpha(1-\varphi\gamma_x) \\ \varphi(1-\gamma_\pi) & (1-\varphi\gamma_x) \end{bmatrix} \tag{3.17}$$

[8] The policy rule (2.11) is just an element of the set of expectations-based policy rule, with:

$$\gamma = \gamma^*, \gamma_x = \gamma_x^*, \gamma_\pi = \gamma_\pi^*, \gamma_u = \gamma_u^*, \gamma_g = \gamma_g^*$$

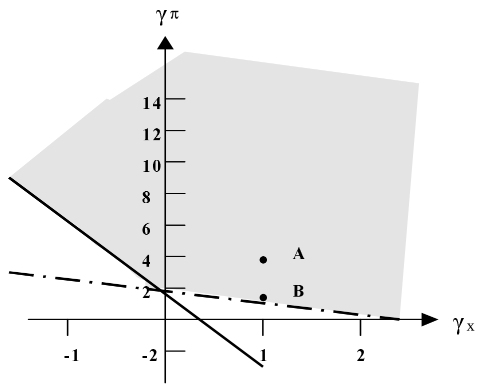
and $\widehat{E}_t Y_{t+1}$ are the expected values of inflation and output gap, determined using least square learning, as described in (3.10), (3.11), (3.12) and (3.13). The ALM of inflation and output gap could be written as

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = Q + R \times \begin{bmatrix} a_{\pi,t} \\ a_{x,t} \end{bmatrix} + \left(\rho_u R \times \begin{bmatrix} b_{\pi,t} \\ b_{x,t} \end{bmatrix} + S \right) u_t \tag{3.18}$$

In order to have E-Stability of the REE we need **all the eigenvalues of the matrixes R and $\rho_u R$ to have real parts less than one**, otherwise the rational expectation equilibrium will not be learnable[9].

In the following pictures I show all the reaction functions under which we have E-stability of the REE; all the combination γ_π and γ_x inside the shadowed area determine that all the eigenvalues of the matrixes R have real parts less than one. The two pictures differ for the values of the parameters: the first one considers the Clarida, Gali and Gertler (1999) parametrization, while the second one is derived with the Woodford (1999)[10] parameters value.

*Fig.*1 The E-stable region under the expectations-based policy rule (CGG parametrization)

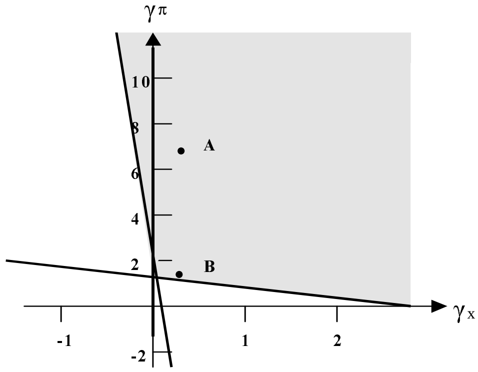


[9] In the Appendix 1, I show how to determine the necessary and sufficient conditions for E-Stability of the rational expectation equilibrium.

[10] Clarida, Galí and Gertler (CGG) and Woodford (W) derive from regressions on US data respectively, the following values for the economy parameters

$$\begin{aligned} \varphi &= 1, \ \ \alpha = 0.3, \ \ \beta = 0.9, \ \ \rho_u = 0.35 \\ \varphi &= 6.37, \ \ \alpha = 0.024, \ \ \beta = 0.99, \ \ \rho_u = 0.35 \end{aligned}$$

*Fig.*2 The E-stable region under the expectations-based policy rule (Woodford parametrization)



It is interesting to represent also the *specific expectations-based* policy rule (2.11); the points A and B in the two pictures correspond to the combination $\gamma_{\pi}^{\star}, \gamma_{x}^{\star}$ in the two extreme cases where the policy maker does not care about the output gap, $\lambda = 0$ (point A) and where he gives equal weight to both inflation and output gap, $\lambda = 1$ (point B)[11]. The pictures show that in both cases the rational expectation equilibrium is E-Stable. It is possible, moreover, to show that for any positive and finite value of λ, i.e. for all *flexible inflation targeting* policies, under the *expectation-based* reaction function (2.11) the rational expectation equilibrium is E-Stable (Evans and Honkapohja (2001)).

3.4 Optimality and Learning

Evans and Honkapohja (2002) say that the *expectation-based* reaction function (2.11) is not only a "good" policy because it determines an E-stable REE, but moreover it "*implements optimal discretionary policy in every period and for all values of private expectations*" also in a context where "*private agents behave in a boundedly rational way*". In order to identify (2.11) as the optimal policy rule under discretion and learning it is crucial here the assumption that "*the policy maker does not make active use of learning behavior on the part of agents*" (Evans and Honkapojha, 2002).

However, if under rational expectation the problem of optimal "discretionary policy" implies by definition that policy maker cannot affect private agents expectations, under the hypothesis of bounded rational private agents, since policy decisions actually do affect the learning process, a perfectly rational policy maker should take it into account in solving the monetary policy design problem. In fact, if private agents' expectations are the result of the estimates of the learning parameters that depend on the past values of the monetary policy instrument, the policy maker, through his decisions, will affect the estimates

[11] In the literature, the case $\lambda = 0$, is defined as *strict inflation targeting policy*, while, in general when $\lambda > 0$, we talk about *flexible inflation targeting policy*

and, cosequently the learning process of private agents (at least in the transition to the REE).

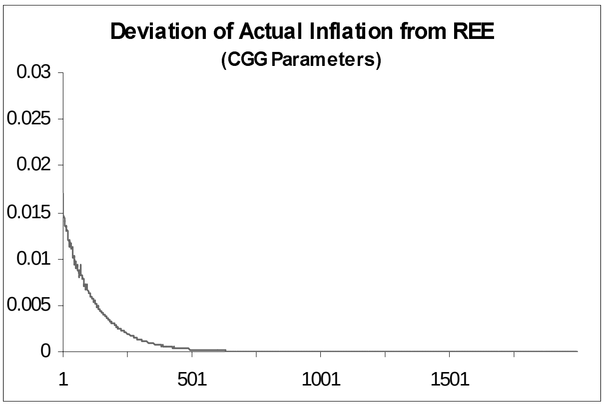
In this case, the *expectation-based* reaction function (2.11) is not necessarily optimal under learning, but could be defined as asymptotically-optimal: if private agents' perceived law of motion is well specified, once that the learning process has converged to rational expectations, than we know that policy rule (2.11) will be optimal. Since there is an infinite number of *Expectations-based policy rule* that, once the agents have learned, determines the same optimal REE, but in the transition to the REE each policy will determine a different dynamic in the economy, than the problem of discriminate between them becomes relevant. We will see in the rest of the paper how this aspect has to be taken into account in the monetary policy design problem.

3.5 The Transition to the REE

In the preceding section we have seen that the expectations-based policy rule (2.11) derived as the optimal solution of the problem under discretion and rational expectation could be taken as a starting point in the analysis of the monetary policy design problem under learning. We have seen that since this policy determines a unique and E-stable REE it is a "good" policy[12] not only under RE but even under boundedly rational private agents hypothesis.

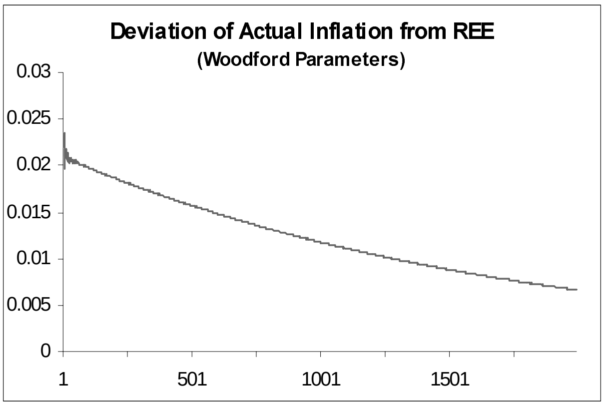
However, if we simulate such an economy, we observe that private agents would need many periods in order to learn the REE. The following pictures show that, under the two different parametrizations and assuming a flexible *inflation targeting policy* with $\lambda = 0.5$, with an initial error in expected inflation of 1,5 percentage points, we need, in order to converge to the REE, more than 500 periods (quarters) under the CGG parametrization and more than 2000 quarters under the Woodford parametrization!

*Fig.*3 Deviation of actual inflation from the REE (CGG parametrization)



[12]Here with "good" policy I refer to Bullard and Mitra (2001) criterion for discriminate between policy rules. They suggest that a good policy is one under which we have both determinacy and learnability of the rational expectations equilibria.

*Fig.*4 Deviation of actual inflation from the REE (Woodford parametrization)



These pictures show that learnability is not by itself sufficient to characterize a policy in a context of bounded rationality. The fact that the learning speed could be very slow means that when he considers the monetary policy design problem under learning, the policy maker should care about the transition to the REE.

4 Speed of Convergence to the REE

In order to understand why the learning process could be very slow, we investigate what determines the speed of convergence of the learning process to the REE. The pictures (Fig. 1 and Fig. 2) that show the combinations of γ_π and γ_x for which we have E-stability, suggest the reason why the learning process could be very slow; for values of λ equal or bigger than one, we are at the border of the E-stability region, that is the eigenvalues of the transition matrices are near to unit. Intuitively, the coefficients in the reaction function are a component that determines the eigenvalues of the transition matrix; the eigenvalues of the transition matrix determine the speed at which agents learn.

In the literature, the problem of the speed of convergence of recursive least square learning algorithms has been analyzed mainly through numerical procedures and simulations. The few analytical results (Marcet and Sargent (1992)) are obtained by using a theorem of Benveniste, Metiver and Priouret (1990). Using this theorem, it is possible to relate the speed of convergence of the learning process to the eigenvalues of the associated O.D.E. at the fixed point. In our case the eigenvalues are the ones of the matrices R and $\rho_u R$ in the system

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = Q + R \times \begin{bmatrix} a_{\pi,t} \\ a_{x,t} \end{bmatrix} + \left(\rho_u R \times \begin{bmatrix} b_{\pi,t} \\ b_{x,t} \end{bmatrix} + C \right) u_t \tag{4.1}$$

In particular, it could be shown through simulations that **the bigger the eigenvalues, the slower the learning process**. As special cases we could consider two examples: *E-stability* and *Root-t Convergence*:

1. If all the eigenvalues of the matrixes R and $\rho_u R$ have real parts bigger than one, as we have already seen, the REE is not learnable (i.e., not E-stable).

2. If all the eigenvalues of the matrixes R and $\rho_u R$ have real parts less than one half, then we have Root-t convergence, that is

$$\sqrt{t}\left(\left[\begin{array}{c} a_{\pi,t} \\ a_{x,t} \end{array}\right]-\left[\begin{array}{c} a_{\pi} \\ a_{x} \end{array}\right]\right)\xrightarrow{D} N\left(0,P_a\right) \tag{4.2}$$
$$\sqrt{t}\left(\left[\begin{array}{c} b_{\pi,t} \\ b_{x,t} \end{array}\right]-\left[\begin{array}{c} b_{\pi} \\ b_{x} \end{array}\right]\right)\xrightarrow{D} N\left(0,P_b\right)$$

   This means that, if the conditions of the theorem are satisfied, the speed at which the estimates $a_{\pi,t}$, $a_{x,t}$, $b_{\pi,t}$, $b_{x,t}$, converge to the true values a_{π}, a_x, b_{π}, b_x, is root-t, that is the speed at which in classical econometrics, the mean of the distribution of the Least Square Estimates converge to the true values of the parameters estimated[13].

4.1 Speed of Convergence and Policy Design

In terms of policy decisions, we have seen that the policy maker through his reaction function will affect the values of the coefficients of the matrixes R and $\rho_u R$. This means that the evolution of estimated coefficients used in the forecasting process (i.e., the speed at which private agents learn) strictly depends on the policy that has been implemented.

We could represent graphically all these informations plotting the speed of convergence level curves. The level curves represent combinations of the policy coefficients γ_{π} and γ_x of a generic *expectations-based reaction function* (2.20) that determine a given speed of convergence, under the parameterization of Clarida, Gali and Gertler (1999) (fig. 5) and by Woodford (1999) (fig.6).

[13] Marcet and Sargent show an important consequence of this theorem, that is, for eigenvalues of the transition matrices higher than 0.5, convergence is slower in the sense that the asymptotic variance-covariance matrices (P_a,P_b) of the limiting distribution are bigger. In particular, what happens is that the importance of initial conditions fails to die out at an exponential rate (as is needed for *root-t convergence*).

Fig.5 The speed of learning isoquants under the *expectations-based policy rule* (CGG parametrization)

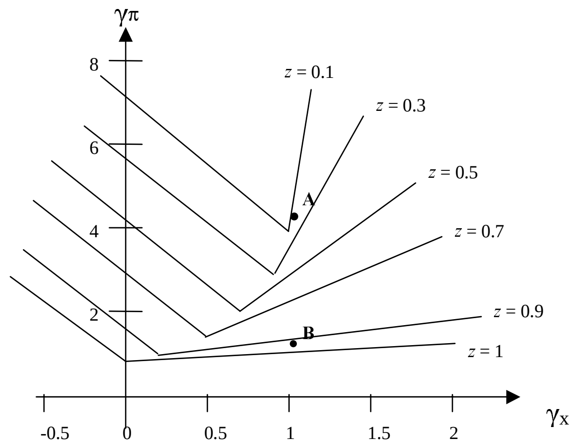
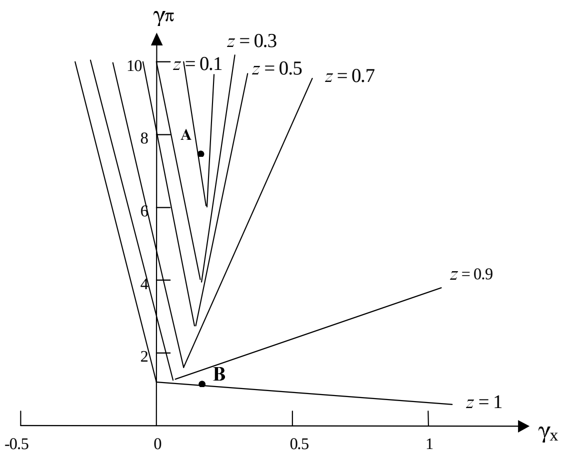


Fig.6 The speed of learning isoquants under the *expectations-based policy rule* (Woodford parameters)



Result 1: *The level curves could be used to discriminate between policies that deliver different speed of the transition to the REE.*

Each level curve corresponds to the combination of policy parameters γ_π and γ_x that determines the same eigenvalues of the matrix R[14]. For example, the combinations of γ_π and γ_x that stay below the level curve $z = 1$ implies an explosive learning process: for these combinations of the policy coefficients,

[14] Here we consider only the matrix R, because if the eigenvalues of R are lower than a certain value x, even the matrix $\rho_u R$ will have eigenvalues smaller than x. Moreover, notice that the two matrices have two eigenvalues, but since in order to have a certain velocity of convergence we need all the eigenvalues to be lower than a certain level, we just consider the bigger of the two. For example, in orther to have root-t convergence we need al the eigenvalues of the $\rho_u R$ and R to have the real part smaller than 0.5, that means a necessary and sufficient condition in order to have root-t convergence we need the bigger eigenvalue of the matrix R to have the real part smaller than 0.5.

we don't have E-stability. The combinations of γ_π and γ_x that stay above the level curve $z = 0.1$ implies a very fast learning process. The combinations of γ_π and γ_x that stay above the level curve $z = 0.5$ implies a learning process that converges to the REE at a *root-t* speed.

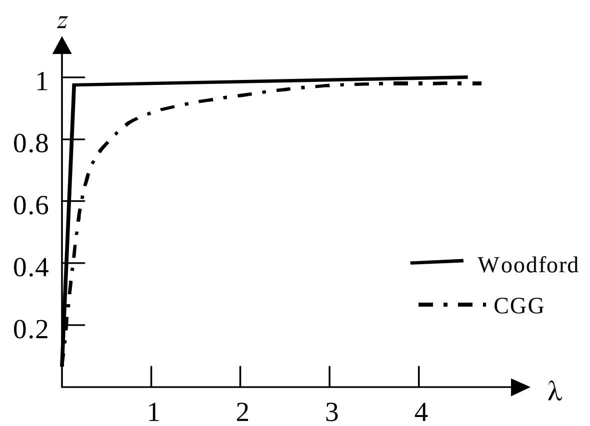
Result 2: *The speed of convergence of the learning process under the optimal expectations-based reaction function (2.11), depends negatively on the weight that the policy maker gives to output gap relatively to inflation. With flexible inflation targeting policies the private agents' learning process is very slow. With strict inflation targeting policies the private agents' learning process is very fast.*

The literature on monetary policy under rational expectation have shown that under an inflation targeting policy, the bigger the weight that the policy maker gives to output gap, the slower will be the convergence to the inflation target.

In our case, under bounded rationality, if we consider the specific expectations-based policy rule (2.11), derived without taking into account that the policy maker could make active use of private agents' learning behavior, we find an interesting result that relates speed of convergence and relevance of the output gap objective. In figures 5 and 6 we can see that, under both parametrization of CGG and Woodford, when the policymaker cares equally about output gap and inflation ($\lambda = 1$), the value of $\gamma_\pi^\star, \gamma_x^\star$ (point B) is on the level curve $z = 0.9$ (i.e., the biggest eigenvalue of the matrix R is almost 0.9); in this case the learning process is very slow. When the policy make follow a strict inflation targeting policy ($\lambda = 0$), instead, the value of $\gamma_\pi^\star, \gamma_x^\star$ (point A) is on the level curve $z = 0.1$ (i.e., the biggest eigenvalue of the matrix R is almost 0.1); in this case we have a very fast learning process.

To have a clearer picture of how, under the optimal expectations-based policy derived without taking into account that the policy maker could make active use of private agents' learning behavior, the biggest eigenvalue of the matrix R changes as the parameter λ increases, let's consider the following picture.

*Fig.*7 Speed of convergence under the *expectations-based policy rule* for different λ's



Result 3: *The speed of convergence of the learning process under a restricted set of asymptotically-optimal expectations-based reaction functions, depends negatively on the weight that the policy maker gives to output gap relatively to inflation.*

Let's consider a subset of asymptotically-optimal expectations-based reaction functions:

$$i_t = \gamma + \gamma_x \widehat{E}_t x_{t+1} + \gamma_\pi \widehat{E}_t \pi_{t+1} + \gamma_u u_t + \gamma_g g_t$$

with

$$\begin{aligned} \gamma_g &= \gamma_g^R \\ \gamma &= \gamma^* \\ \gamma_\pi &= \frac{\gamma^R - \gamma^*}{a_\pi} - \frac{a_x}{a_\pi}\gamma_x \\ \gamma_u &= \gamma_u^R - \frac{\left(\gamma^R - \gamma^*\right) b_\pi \rho_u}{a_\pi} + \gamma_x \rho_u \left(\frac{a_x}{a_\pi} b_\pi - b_x\right) \end{aligned} \tag{4.3}$$

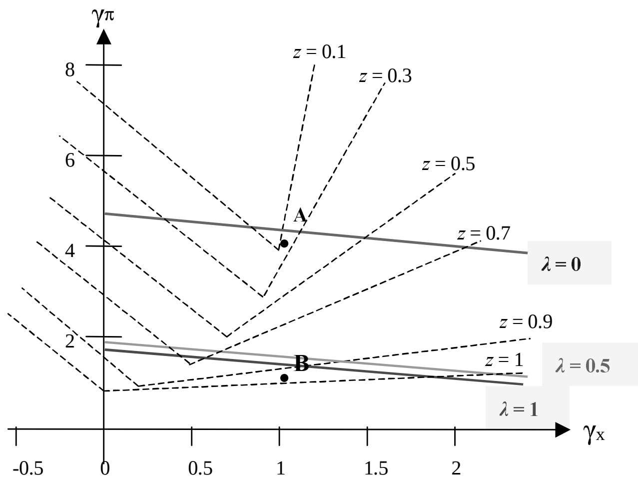
These policies, once the private agents' learning process has converged to rational expectations, will determine the same equilibrium that we would have with the optimal monetary policy under discretion and rational expectations. Moreover, in this reaction function the coefficient of the constant term γ is equal to the one of the expectations-based reaction function (2.11).

Substituting the values of γ_g^R, γ_u^R, γ^R, γ^* determined in equation (2.12) and (2.19) into the system of equations (4.3), we obtain the combination of γ_x and γ_π that determine asyntotically the same equilibrium that we obtain under (2.11):

$$\gamma_\pi = \frac{\left(\lambda + \alpha^2\right)(1 + \alpha\varphi) - \lambda\beta}{\alpha\left(\lambda + \alpha^2\right)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x$$

Graphically, we obtain the following mapping, under CGG parametrization (a similar result could be obtained under Woodford parametrization):

*Fig.*8 A subset of asymptotically-optimal expectations based reaction functions



From this picture we can see that under an expectations-based reaction function, that is asymptotically optimal and $\gamma = \gamma^{*}$, the higher is the relative weight on output gap (λ), the lower will be the maximum speed of convergence of the learning process that a policymaker could induce. In particular, in this case, with $\lambda \succeq 0.5$ we have no combination of $\gamma_{\pi}^{\star}, \gamma_{x}^{\star}$ that could determine at least a root-t convergence.

The same result is obtained with a different subset of a*symptotically-optimal expectations-based* reaction functions, characterized by thefollowing coefficients

$$\begin{aligned}
\gamma_g &= \gamma_g^R \\
\gamma_u &= \gamma_u^* \\
\gamma_\pi &= \frac{\left(\gamma_u^R - \gamma_u^*\right)}{b_\pi \rho_u} - \frac{b_x}{b_\pi}\gamma_x \\
\gamma &= \gamma^R - \frac{\left(\gamma_u^R - \gamma_u^*\right) a_\pi}{b_\pi \rho_u} + \left(\frac{b_x}{b_\pi}a_\pi - a_x\right)\gamma_x
\end{aligned} \tag{4.4}$$

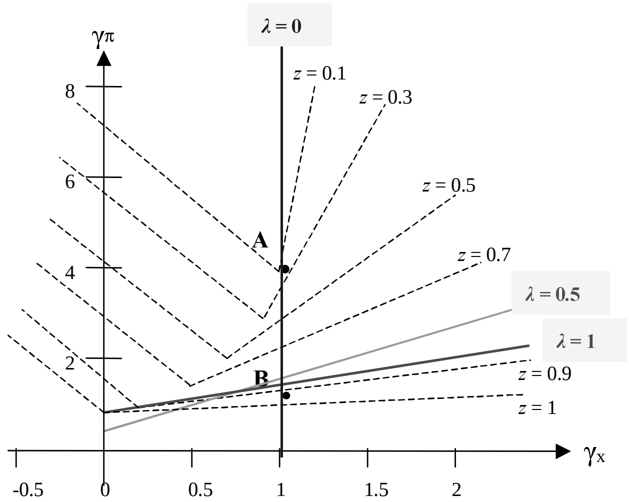
This policy, once the private agents' learning process has converged to rational expectations, will determine the same equilibrium that we would have with the previous subset of policies and under the policy (2.11). Notice that in this reaction function the coefficient that shows the reaction to cost push shocks (γ_u) is equal to the one of the expectations-based reaction function (2.11).

Substituting the values of γ_g^R, γ_u^R, γ^R, γ_u^* determined in equation (2.12) and (2.19) into the system of equations (4.4), we obtain the combination of γ_x and γ_π that determine asyntotically the same equilibrium that we obtain under (2.11)

$$\gamma_\pi = \frac{\left(\lambda + \alpha^2\right)(\varphi\lambda - \alpha) + \lambda\beta\alpha}{\left(\lambda + \alpha^2\right)\varphi\lambda} + \frac{\alpha}{\lambda}\gamma_x$$

Graphically, we obtain the following mapping, under CGG parametrization (a similar result could be obtained under Woodford parametrization):

*Fig.*9 A subset of asymptotically-optimal expectations based reaction functions



We want now to see how the policy maker can make active use of private agents learning behavior[15].

4.2 The Mapping from PLM to ALM

In the previous sections we have analyzed an economy where agents formulate forecasts through recursive least square regressions, following the traditional approach that looks at the eigenvalues of the transition matrices. In order to have a clearer picture of what happens in such an economy we consider now more in detail the mapping from perceived to actual variables.

Let's define the parameters

$$\begin{aligned} \Gamma^* &= \frac{\lambda\beta}{(\lambda+\alpha^2)} \\ \Phi^* &= \frac{\lambda\alpha}{(\lambda+\alpha^2)}\overline{x} \\ \Psi^* &= \frac{\lambda}{(\lambda+\alpha^2)} \end{aligned} \tag{4.5}$$

and rewrite the *expectation-based reaction function* (2.11) proposed by Evans and Honkapohja (2002) in the following way

$$i_t^{EH} = \gamma^\star + \gamma_x^\star \widehat{E}_t x_{t+1} + \gamma_\pi^\star \widehat{E}_t \pi_{t+1} + \gamma_u^\star u_t + \gamma_g^\star g_t \tag{4.6}$$

[15] In a follow up paper I characterize the speed of convergence of the learning process under alternative policy rules

where the coefficients could be rewritten as

$$\begin{aligned}\gamma^* &= -\frac{\Phi^*}{\varphi\alpha} \qquad (4.7)\\ \gamma_x^*, \gamma_g^* &= \frac{1}{\varphi}\\ \gamma_\pi^* &= \left(1+\frac{\beta-\Gamma^*}{\alpha\varphi}\right)\\ \gamma_u^* &= \frac{(1-\Psi^*)}{\varphi\alpha}\end{aligned}$$

Under this policy rule the economy evolves according to the two following equations:

$$\pi_t = \Phi^* + \Gamma^* \widehat{E}_t \pi_{t+1} + \Psi^* u_t \qquad (4.8)$$

$$x_t = \frac{\Phi^*}{\alpha} - \frac{(\beta-\Gamma^*)}{\alpha} \widehat{E}_t \pi_{t+1} - \frac{(1-\Psi^*)}{\alpha} u_t \qquad (4.9)$$

We know that under least square learning the ALM of π_t and x_t will be

$$\begin{aligned}\pi_t &= \Phi^* + \Gamma^* a_{\pi,t} + \left(\rho_u \Gamma^* b_{\pi,t} + \Psi^*\right) u_t \qquad (4.10)\\ x_t &= \frac{\Phi^*}{\alpha} - \frac{(\beta-\Gamma^*)}{\alpha} a_{\pi,t} + \left(-\rho_u \frac{(\beta-\Gamma^*)}{\alpha} b_{\pi,t} - \frac{(1-\Psi^*)}{\alpha}\right) u_t\end{aligned}$$

In order to analyze convergence of the learning equilibrium to the REE, we could consider the following mappings[16]

$$S_{a_\pi}\left(a_{\pi,t}\right) = \Phi^* + \Gamma^* a_{\pi,t} \qquad (4.11)$$

$$S_{b_\pi}\left(b_{\pi,t}\right) = \Psi^* + \Gamma^* \rho_u b_{\pi,t} \qquad (4.12)$$

$$S_{a_x}\left(a_{\pi,t}, a_{x,t}\right) = \frac{\Phi^*}{\alpha} - \frac{(\beta-\Gamma^*)}{\alpha} a_{\pi,t} \qquad (4.13)$$

$$S_{b_x}\left(b_{\pi,t}, b_{x,t}\right) = -\rho_u \frac{(\beta-\Gamma^*)}{\alpha} b_{\pi,t} - \frac{(1-\Psi^*)}{\alpha} \qquad (4.14)$$

These mappings show, in each period, how the estimates used by private agents in order to make forecasts $(a_{\pi,t}, a_{x,t}, b_{\pi,t}, b_{x,t})$, would affect the actual value of inflation and output gap. Notice that these four expressions could be interpreted as mappings from perceived to actual dynamics of the two variables.

In particular, we could use the following definition:

[16] For details, see Appendix

Definition 1 *Let's consider the four mappings (4.11), (4.12), (4.13), (4.14). If*

$$S_{a_\pi}(a_{\pi,t}) \to a_{\pi,t} = a_\pi = \Phi^* (1-\Gamma^*)^{-1} \tag{4.15}$$

$$S_{b_\pi}(b_{\pi,t}) \to b_{\pi,t} = b_\pi = \Psi^* (1-\rho_u\Gamma^*)^{-1} \tag{4.16}$$

$$S_{a_x}(a_{\pi,t}, a_{x,t}) \to a_{x,t} = a_x = \frac{(1-\beta)}{\alpha}\frac{\Phi^*}{(1-\Gamma^*)} \tag{4.17}$$

$$S_{b_x}(b_{\pi,t}, b_{x,t}) \to b_{x,t} = b_x = \frac{(1-\rho_u\beta)\Psi^* - (1-\rho_u\Gamma^*)}{\alpha(1-\rho_u\Gamma^*)} \tag{4.18}$$

we say that the ***learning process have converged to a stationary equilibrium, the REE****.*

If we consider the first two mappings, we could easily verify that a necessary condition for convergence is that

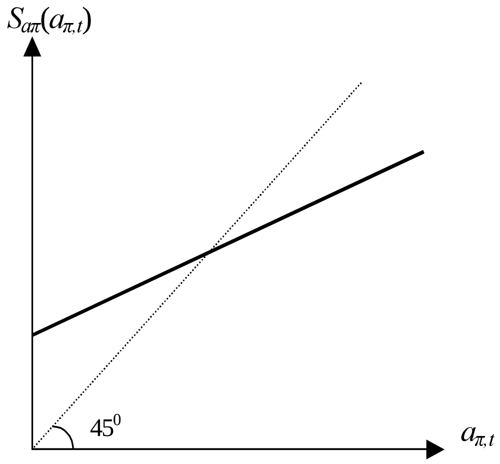
$$\Gamma^* = \frac{\lambda\beta}{(\lambda+\alpha^2)} < 1 \tag{4.19}$$

In this case, the mapping[17]

$$S_{a_\pi}(a_{\pi,t}) = \Phi^* + \Gamma^* a_{\pi,t} \tag{4.20}$$

have a slope smaller than one.

*Fig.*10 The mapping $S_{a_\pi}(a_{\pi,t})$ from PLM to ALM



[17] Similarly for the mapping

$$S_{b_\pi}(b_{\pi,t}) = \Psi^* + \Gamma^*\rho_u b_{\pi,t}$$

the necessary condition for convergence is $\Gamma^*\rho_u < 1$. Notice that if $\Gamma^* < 1$ the condition is obviously satisfied since $\rho_u < 1$

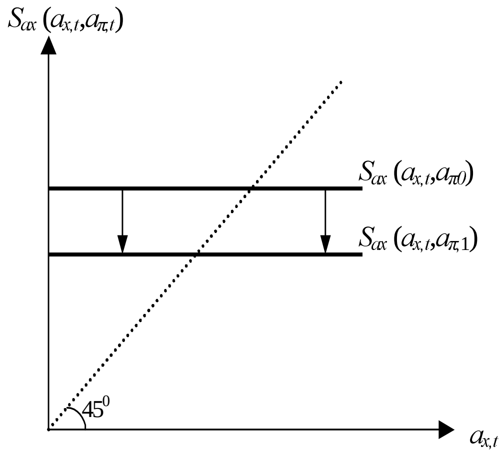
If we consider the third and forth mappings, we could easily verify that they depends only on what happens to the first two. In particular notice that the slope of S_{a_π} and S_{a_x} with respect to $a_{\pi,t}$ and $a_{\pi,t}$ is equal to zero.

Graphically this means that the mapping[18]

$$S_{a_x}\left(a_{\pi,t}, a_{x,t}\right) = \frac{\Phi^*}{\alpha} - \frac{(\beta - \Gamma^*)}{\alpha} a_{\pi,t} \tag{4.21}$$

is an horizontal line

*Fig.*11 The mapping $S_{a_\pi}(a_{\pi,t})$ from PLM to ALM



Notice that, since

$$\frac{\partial S_{a_x}\left(a_{\pi,t}, a_{x,t}\right)}{\partial a_{\pi,t}} < 0 \tag{4.22}$$

if $a_{\pi,t}$ is increasing over the learning process (i.e., we approach the fixed point of the mapping $S_{a_\pi}(a_{\pi,t})$ from below), the $S_{a_x}(a_{\pi,t}, a_{x,t})$ mapping will move downward (as in the picture). This downward movement will end up when $a_{\pi,t}$ converges to a_π.

4.3 Speed of convergence and the Mapping from PLM to ALM

Let's now use this framework in order to analyze the speed of convergence of the learning process to the RE. As we have seen before, the speed at which

[18] Similarly for the mapping

$$S_{b_x}\left(b_{\pi,t}, b_{x,t}\right) = -\rho_u \frac{(\beta - \Gamma^*)}{\alpha} b_{\pi,t} - \frac{(1 - \Psi^*)}{\alpha}$$

agents learn depends on the eigenvalues of the matrix B. That the value of the Eigenvalues, under the *optimal expectations-based policy* that does not take into account the transition to the REE (2.11), are

$$\begin{aligned} z_1 &= 0 \\ z_2 &= \frac{\lambda\beta}{(\lambda+\alpha^2)} = \Gamma^* \end{aligned} \tag{4.23}$$

Notice that Γ^* is also the slope of the mapping $S_{a_\pi}(a_{\pi,t})$.

Since I'm considering λ as an exogenous policy parameter, we can assume that the policy maker gives a positive weight to output gap, for example $\lambda = 0.5$ (notice, that with this weight, we are assuming that the policymaker cares two times more about inflation than about output gap). In this case the mapping

$$S_{a_\pi}(a_{\pi,t}) = \Phi^* + \Gamma^* a_{\pi,t} \tag{4.24}$$

will have a slope equal to 0.76 under CGG parametrization and 0.99 under the Woodford's one; graphically the mapping will be

*Fig.*12 The mapping $S_{a_\pi}(a_{\pi,t})$ from PLM to ALM under CGG parametrization and λ=0.5

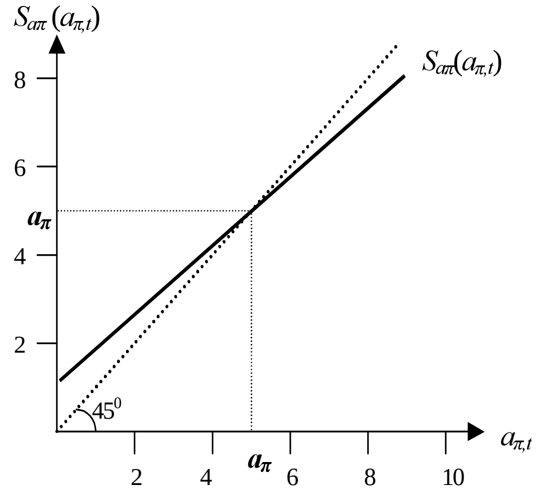
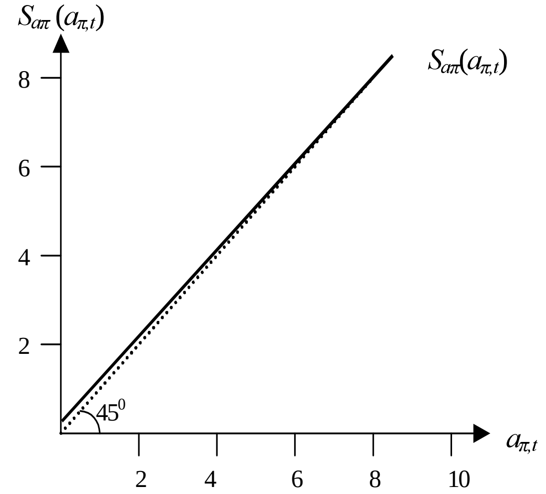


*Fig.*13 The mapping $S_{a_\pi}(a_{\pi,t})$ from PLM to ALM under Woodford parametrization and λ=0.5



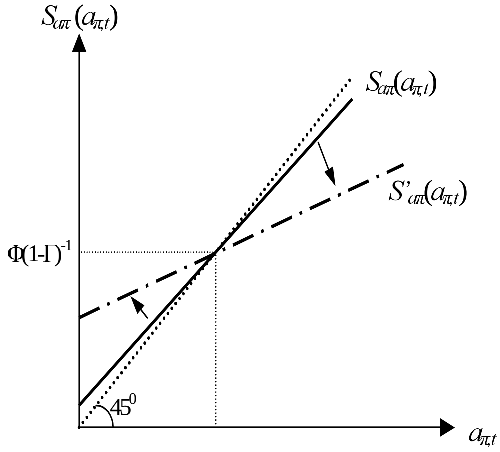
From these pictures (in particular the one that refers to Woodford parametrization) it is easy to see that, in order to converge to the REE (the point where the mapping crosses the 45 degree line), even if we start from a point relatively not too far from the REE, we will need many periods, since the slope of the mapping is close to one.

4.4 Adjusting the learning Speed without affecting the REE

In the previous section, under the specific *expectation-based reaction function* (2.11) suggested by Evans and Honkapohja (EH policy), we saw that if the policy maker want to follow a flexible inflation targeting policy rule, the private agents' learning process will converge very slowly to the RE. Now, we could ask how a policy maker that wants to reach in the long run (i.e., once the private agents have learned the REE) the same REE determined by the reaction function (2.11), could speed-up or slow down the private agents learning process.

In order, for example, to speed-up the transition, the policy maker could follow an Adjusted Learning Speed (ALS) *expectations-based* policy rule that determines the new mappings $S'_{a_\pi}(a_{\pi,t})$ and $S'_{b_\pi}(b_{\pi,t})$. This mappings will have the same fixed points of $S_{a_\pi}(a_{\pi,t})$ and $S_{b_\pi}(b_{\pi,t})$ (i.e., once the learning process has converged to RE, we have a policy that is optimal under discretion and RE), but with a smaller slope Γ'. Graphically, the new policy should generate a rotation of the mapping taking as fixed the point where it crosses the 45 degree line.

*Fig.*14 The mapping $S'_{a_\pi}(a_{\pi,t})$ under the *Adjusted-Learning-Speed policy*



Let's

$$1 > \Gamma^* > \Gamma' \tag{4.25}$$

then the new mapping $S'_{a_\pi}(a_{\pi,t})$ will be[19]

$$S'_{a_\pi}(a_{\pi,t}) = \Phi^* (1-\Gamma^*)^{-1} (1-\Gamma') + \Gamma' a_{\pi,t} \tag{4.26}$$

The resulting value for inflation will be

$$\pi_t = \frac{\Phi^* (1-\Gamma')}{(1-\Gamma^*)} + \Gamma' E_t \pi_{t+1} + \frac{\Psi (1-\rho_u \Gamma')}{(1-\rho_u \Gamma^*)} u_t \tag{4.27}$$

and the one for output gap,

$$x_t = \frac{\Phi^* (1-\Gamma')}{(1-\Gamma^*)\,\alpha} - \frac{(\beta-\Gamma')}{\alpha} E_t \pi_{t+1} - \frac{\left(1-\Psi^* (1-\rho_u \Gamma^*)^{-1} (1-\rho_u \Gamma')\right)}{\alpha} u_t \tag{4.28}$$

and the new *Adjusted-Learning-Speed expectation* reaction function will be:

$$i_t^{ALS} = \gamma' + \gamma'_x \widehat{E}_t x_{t+1} + \gamma'_\pi \widehat{E}_t \pi_{t+1} + \gamma'_u u_t + \gamma'_g g_t \tag{4.29}$$

[19] Similarly, the new mapping $S'_{b_\pi}(b_{\pi,t})$ will be

$$S'_{b_\pi}(b_{\pi,t}) = \Psi^* (1-\rho_u \Gamma^*)^{-1} \left(1-\rho_u \Gamma'\right) + \Gamma' \rho_u b_{\pi,t}$$

where

$$\begin{aligned}
\gamma' &= -\frac{\Phi^* (1-\Gamma^*)^{-1} (1-\Gamma')}{\alpha\varphi} \\
\gamma'_x, \gamma'_g &= \frac{1}{\varphi} \\
\gamma'_\pi &= \left(1 + \frac{(\beta - \Gamma')}{\alpha\varphi}\right) \\
\gamma'_u &= \frac{\left(1 - \Psi^* (1-\rho_u \Gamma^*)^{-1} (1 - \rho_u \Gamma')\right)}{\varphi\alpha}
\end{aligned} \tag{4.29}$$

Comparing now the two policies we obtain the following results:

Result 4: *If we impose*

$$\Gamma' = \Gamma^* \tag{4.30}$$

the EH and the ALS expectation-based policies will coincide.

Just substitute (4.30) into (4.29) and we obtain (4.6).

Result 5: *When the learning process has converged to the rational expectation equilibrium, the level of inflation and output gap*[20] *is the same under the EH and the ALS expectation-based policies:*

$$\pi_t = \Phi^* (1-\Gamma^*)^{-1} + \Psi^* (1-\rho_u\Gamma^*)^{-1} u_t \tag{4.31}$$

$$x_t = \frac{(1-\beta)}{\alpha} \frac{\Phi^*}{(1-\Gamma^*)} - \frac{(1-\rho_u\Gamma^*) - (1-\beta\rho_u)\Psi^*}{\alpha(1-\rho_u\Gamma^*)} u_t \tag{4.32}$$

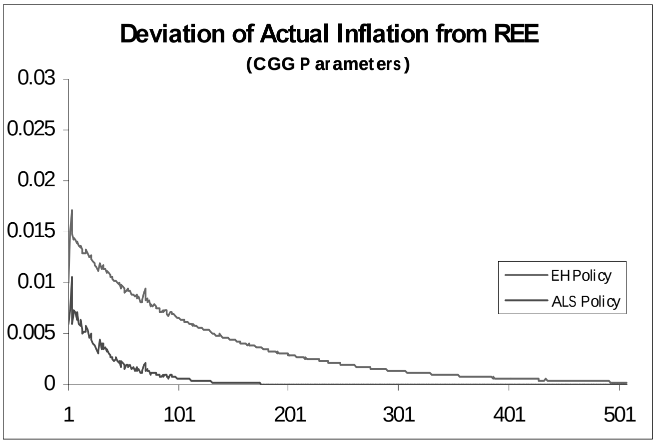
Proof in the Appendix.

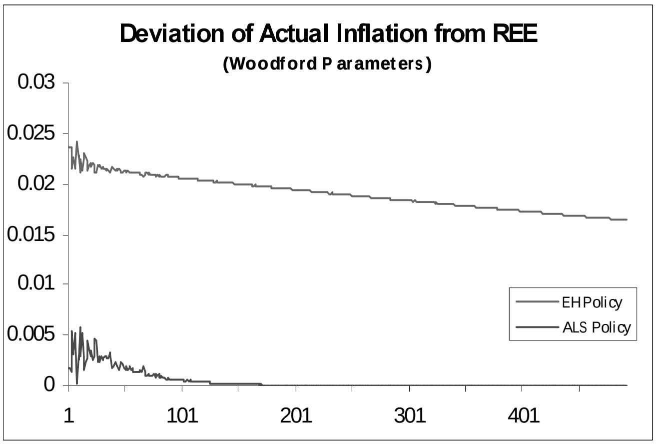
Result 6: *If the economy starts from a point where the learning equilibrium and the REE do not coincide, than the transition to the REE will be shorter under the ALS policy than under the EH policy rule.*

Let's consider again a simulation where the initial expectation error under both policies is 10%, and use a value of $\Gamma' = 0.5$ (i.e., we impose root-t convergence). The results of the simulations are in the following pictures and show that under the ASL policy the transition is at least five times shorter than under the EH policy.

[20] It is easy to show that both policy rules will be

$$i_t = \Phi^* (1-\Gamma^*)^{-1} + \frac{\varphi\alpha\Psi^*\rho_u + (1-\rho_u\Gamma^* - (1-\beta\rho_u)\Psi^*)(1-\rho_u)}{\varphi\alpha(1-\rho_u\Gamma^*)} u_t + \frac{1}{\varphi} g_t$$





Result 7: *In the transition to the REE*[21]*, under the ALS policy rule, the policy maker increases the nominal interest rate by a larger amount in response to a rise in Expected Inflation relative to the case of EH policy rule.*

In fact it is easy to show that if

$$1 > \Gamma^* > \Gamma' \tag{4.33}$$

[21] if

$$1 > \Gamma^* > \Gamma'$$

we will have

$$\begin{array}{c} |\gamma_2'| > |\gamma_2^*| \\ \gamma_3' = \gamma_3^* \\ \gamma_4' = \gamma_4^* \\ \gamma_0' > \gamma_0^* \\ \gamma_1' < \gamma_1^* \end{array}$$

than

$$\gamma'_\pi = \left(1 + \frac{(\beta - \Gamma')}{\alpha\varphi}\right) > \left(1 + \frac{\beta - \Gamma^*}{\alpha\varphi}\right) = \gamma^*_\pi \tag{4.34}$$

The intuition is the following: if a policy maker react strongly to a change in expected inflation, private agents learn faster; since at the beginning, if private agents learn faster they would make bigger errors, they will adapt their estimates faster and the transition to the REE will be shorter.

In other words, the stronger the policy maker responds to a change in private agents' expectations, the faster private agents learn and the shorter the transition to the REE.

Result 8: *In the transition to the REE, under the ALS policy the response of inflation to a positive cost-push shock is lower if we start from an expected inflation higher than the REE and is higher than under the EH policy rule if we start from an expected inflation lower than the REE.*

The proof is in the Appendix.

This result implies that

$$\Delta\pi_t^{ALS} > \Delta\pi_t^{EH} \quad \text{IF} \quad b_{\pi,t} < b_\pi \tag{4.35}$$

while

$$\Delta\pi_t^{ALS} < \Delta\pi_t^{EH} \quad \text{IF} \quad b_{\pi,t} > b_\pi \tag{4.36}$$

Result 9: *In the transition to the REE, under the ALS policy rule the response of output gap to a positive cost-push shock is lower if we start from an expected inflation higher than the REE and is higher than under the EH policy rule if we start from an expected inflation lower than the REE.*

The proof is in the Appendix.

This result implies that

$$\Delta x_t^{ALS} > \Delta x_t^{EH} \quad \text{IF} \quad b_{\pi,t} < b_\pi \tag{4.37}$$

while

$$\Delta x_t^{ALS} < \Delta x_t^{EH} \quad \text{IF} \quad b_{\pi,t} > b_\pi \tag{4.38}$$

Result 10: Welfare analysis. *If inflation expectations are higher than the REE, a policy that determines a fast learning transition will increase social welfare with respect to a policy that determines a slow learning process. If inflation expectations are lower than the REE, a policy that determines a slow learning transition will increase social welfare with respect to a policy that determines a fast learning process.*

The policy maker should distinguish between two situations:

1. When inflation expectations are lower than the REE, it turns out that actual inflation under learning will be smaller than actual inflation under RE (and output gap under learning will be higher than output gap under RE). This means that if we look at the policy maker welfare function, it would be preferable that agents learn inflation behavior slowly.

2. When inflation expectations are higher than the REE, actual inflation under learning will be higher than actual inflation under RE (and output gap under learning will be lower than output gap under RE). This means that the policy maker would prefer private agents to learn quickly.

*Tav.*1 Loss in welfare in the first 1000 periods with initial expectations higher than REE

EH	$\Gamma' = 0.01$	$\Gamma' = 0.3$	$\Gamma' = 0.5$	$\Gamma' = 0.7$	$\Gamma' = 0.9$
32%	14%	17%	20%	26%	37%

Table 1 shows the level of welfare under different policies when the initial expected inflation is higher than the REE. The first column shows what happen under the optimal expectation-based policy derived without taking into account that the policy maker could make active use of private agents' learning behavior (the *EH expectation-based policy*); in this case the loss in terms of welfare with respect to the RE case is in the order of 32%. The other columns consider policies adjusted for specific levels of the learning speed: the faster the speed (first columns) the lower the welfare loss.

*Tav.*2 Gain in welfare in the first 1000 periods with initial expectations lower than REE

EH	$\Gamma' = 0.01$	$\Gamma' = 0.3$	$\Gamma' = 0.5$	$\Gamma' = 0.7$	$\Gamma' = 0.9$
23%	11%	14%	17%	21%	24%

If we consider the case where initial expected inflation is lower than the REE (table 2), the welfare gain under learning are determined in the following way: under the optimal expectation-based policy derived without taking into account that the policy maker could make active use of private agents' learning behavior the gain in terms of welfare with respect to the RE case is in the order of 23%. The other columns show that the faster the speed (first columns) the lower the welfare gain.

5 Conclusions and Extensions

In this paper I have shown that considering learning in a model of monetary policy design is particularly important in order to describe not only the rational expectation equilibrium to which we could converge, but even to describe the dynamics that characterize the transition to this equilibrium.

The central message of the paper is that policy makers should look not only at monetary policies that determines a stable equilibrium under learning, but should take into account also how policy decisions affect the speed at which lerning converges to rational expectations. In particular, it is important to know that under certain policies, the REE is E-stable, but the period needed in order to converge to this equilibrium could be incredibly long.

A policy maker that consider his role in determining the dynamics of the private agents learning process, could choose a policy rule that induces agents to learn at a given speed, affecting the welfare of the society.

My current research aims at analyzing the dynamics of the speed of convergence of the learnining process under different policy rules and in the case of monetary policy decisions under commitment.

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6 Appendix

6.1 E-Stability of the REE

In order to derive the ALM of inflation and output gap, we need just to substitute the generic *expectations-based* reaction function

$$i_t = \gamma + \gamma_x \widehat{E}_t x_{t+1} + \gamma_\pi \widehat{E}_t \pi_{t+1} + \gamma_u u_t + \gamma_g g_t$$

into the IS and AS relation. We obtain the dynamic system

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = Q + R \begin{bmatrix} \widehat{E}_t \pi_{t+1} \\ \widehat{E}_t x_{t+1} \end{bmatrix} + S u_t$$

where and

$$Q = \begin{bmatrix} -\alpha\varphi\gamma \\ -\varphi\gamma \end{bmatrix}$$

$$R = \begin{bmatrix} (\beta + \alpha\varphi(1-\gamma_\pi)) & \alpha(1-\varphi\gamma_x) \\ \varphi(1-\gamma_\pi) & (1-\varphi\gamma_x) \end{bmatrix}$$

$$S = \begin{bmatrix} (1-\alpha\gamma_u) \\ -\gamma_u \end{bmatrix}$$

The PLM of the boundedly rational agents is assumed to be well specified[22]. Under least square learning, agents at time t estimate the model

$$\begin{aligned} \pi_t &= a_\pi + b_\pi u_t + \kappa_{\pi t} \\ x_t &= a_x + b_x u_t + \kappa_{xt} \end{aligned}$$

by running a least squares regression of π_t and x_t on an intercept and u_t using data available. Let $(a_{\pi,t},\, b_{\pi,t},\, a_{x,t},\, b_{x,t})$ denote the least squares estimate using data on π_i, x_i and u_i, $i = 1, ..., t-1$. Expectations are then given by

$$\begin{bmatrix} \widehat{E}_t \pi_{t+1} \\ \widehat{E}_t x_{t+1} \end{bmatrix} = A_t + \rho_u B_t u_t$$

[22] A well specified PLM is the one that consider all the state variables that we have in the REE:

$$\begin{aligned} \pi_t &= a_\pi + b_\pi u_t \\ x_t &= a_x + b_x u_t \\ u_t &= \rho_u u_{t-1} + \varepsilon_{u,t} \end{aligned}$$

where, for simplicity, we treat ρ_u as known[23] and where

$$A_t = \left[\begin{array}{c} a_{\pi,t} \\ a_{x,t} \end{array}\right]$$
$$B_t = \left[\begin{array}{c} b_{\pi,t} \\ b_{x,t} \end{array}\right]$$

Then, the ALM of inflation and output gap is

$$\left[\begin{array}{c} \pi_t \\ x_t \end{array}\right] = Q + RA_t + (\rho_u RB_t + S)\, u_t$$

Thus the Mapping from PLM to ALM takes the form

$$T_a\left(A_t'\right) = Q + RA_t$$

and

$$T_b\left(B_t'\right) = \rho_u RB_t + S$$

We consider the stability under learning (E-stability) of the rational expectation solution $(\overline{A}, \overline{B})$, as the situation where the estimated parameters (A_t, B_t) converges to $(\overline{A}, \overline{B})$ over time.

Let's consider the mapping from PLM to ALM:

$$T(T_a, T_b) = (Q + RA, \rho_u RB + S)$$

the Expectation Stability is determined by the following matrix differential equation

$$\frac{d}{d\tau}\left(A', B'\right) = T\left(A', B'\right) - \left(A', B'\right)$$

For this framework E-stability conditions are readily obtained by computing the derivative of $T\left(A', B'\right) - \left(A', B'\right)$ and imposing that the determinant of the matrix with the derivatives of the previous differential equation with respect to A and K is bigger than zero. In particular, we need the eigenvalues of both R and $R\rho_u$, to have real parts less than one.

The eigenvalues of $R\rho_u$ are given by the product of the eigenvalues of R and ρ_u, and since $0 < |\rho_u| < 1$ it suffices that the eigenvalues of R have the real part less than one.

Than, let distinguish again between the two cases:

1. The "Real" Case.

   In this case we need two conditions to be satisfied in order to have convergence to Rational Expectations under Least Squares Learning:

[23] This means that we assume, for simplicity, that private agents do not learn about tthe behavior of the cost push shocks. If not, it too can be estimated by a separate regression of u_t on u_{t-1}.

(a) For Reality

$$\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)^{2}-4\beta\left(1-\varphi\gamma_{x}\right)>0$$

(b) $\boxed{z_1<1}$
That is

$$\frac{\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)}{2}+\frac{\sqrt{\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)^{2}-4\beta\left(1-\varphi\gamma_{x}\right)}}{2}<1$$

Under equality we have

$$\gamma_{\pi}=1-\frac{(1-\beta)}{\alpha}\gamma_{x}$$

Notice, that if $z_1$is smaller than 1 than even z_2 is smaller than 1.

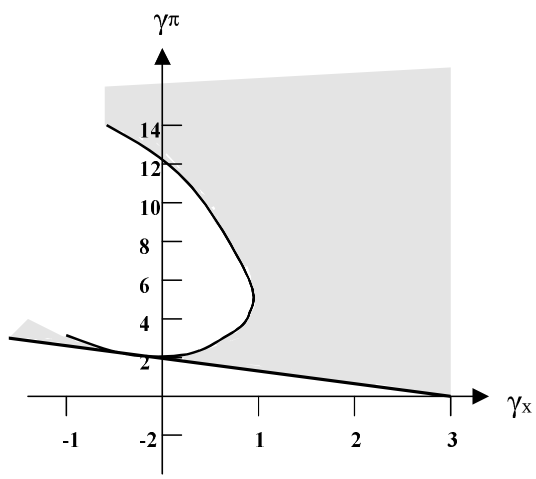
If we assume the values of the parameters of Clarida, Gali and Gertler (2000), the relation

$$1.24-1.32\gamma_{\pi}+0.09\gamma_{\pi}^{2}-0.8\gamma_{x}+\gamma_{x}^{2}+0.6\gamma_{\pi}\gamma_{x}>0$$

shows the combinations of γ_{π},γ_{x} for which the eigenvalues z_1 and z_2 are real.
In order to have $\boxed{z_1<1}$ we need

$$\gamma_{\pi}>1-\frac{1}{3}\gamma_{x}$$

Grafically, we need to be **inside** the shadowed area.



If we assume the values of the parameters from Woodford (1999), the relation

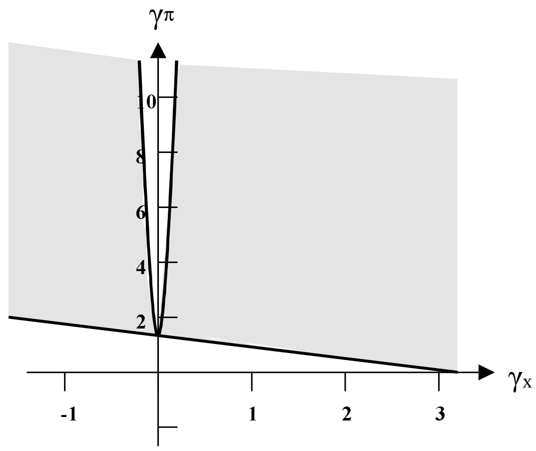
$$(0.15\,(1-\gamma_\pi)+(0.99)+(1-(6.37)\,\gamma_x))^2-3.96\,(1-(6.37)\,\gamma_x)>0$$

shows the combinations of γ_π, γ_x for which the eigenvalues z_1 and z_2 are real.

In order to have $\boxed{z_1<1}$ we need

$$\gamma_\pi>1-0.42\gamma_x$$

Graphically, we need to be **inside** the shadowed area.



2. The "Complex Case".

   In this case we need two conditions to be satisfied in order to have convergence to Rational Expectations under Least Squares Learning:

   (a) For the solution to be imaginary, we need

   $$(\alpha\varphi\,(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta\,(1-\varphi\gamma_x)<0$$

   (b) $\boxed{\text{Real part of } z_1<1}$

   $$\frac{(\alpha\varphi\,(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2}<1$$

   That is

   $$\gamma_\pi>1-\frac{1-\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}$$

   Notice, that if $z_1$is smaller than 1 than even z_2 is smaller than 1.

If we assume that the Clarida, Galí and Gertler's parameters, the relation

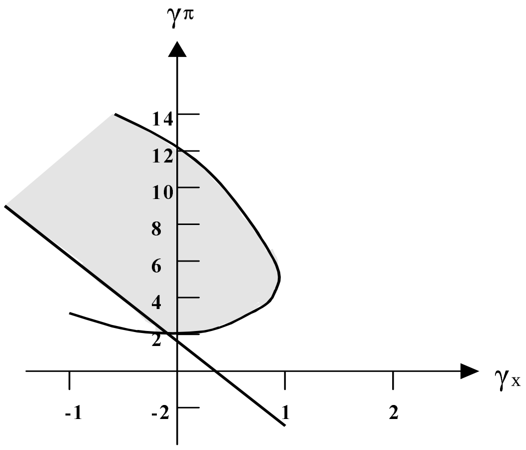
$$(0.3\,(1-\gamma_{\pi})+0.9+(1-\gamma_{x}))^{2}-4*0.9\,(1-\gamma_{x})<0$$

shows the combinations of γ_{π},γ_{x} for which the eigenvalues z_1 and z_2 are complex.

In order to have the eigenvalue z_1 inside the unit circle, we need

$$\gamma_{\pi}>\frac{2}{3}-\frac{10}{3}\gamma_{x}$$

Graphically, we need to be **inside** the shadowed area.



If we assume that the Woodford's parameters, the relation

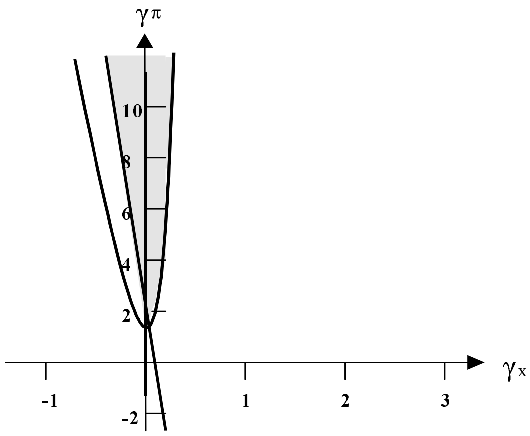
$$(0.15\,(1-\gamma_{\pi})+0.99+(1-(6.37)\,\gamma_{x}))^{2}-3.96\,(1-(6.37)\,\gamma_{x})<0$$

shows the combinations of γ_{π},γ_{x} for which the eigenvalues z_1 and z_2 are complex.

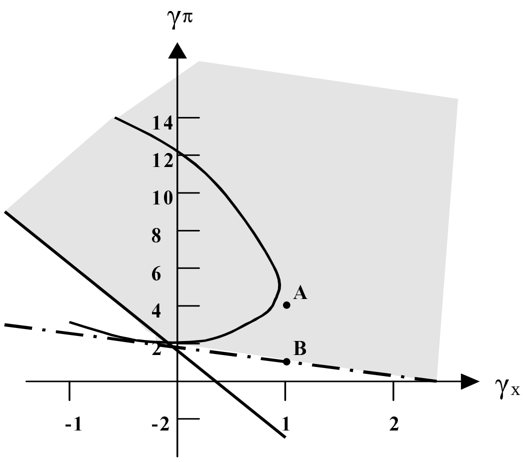
In order to have the eigenvalue z_1 inside the unit circle, we need

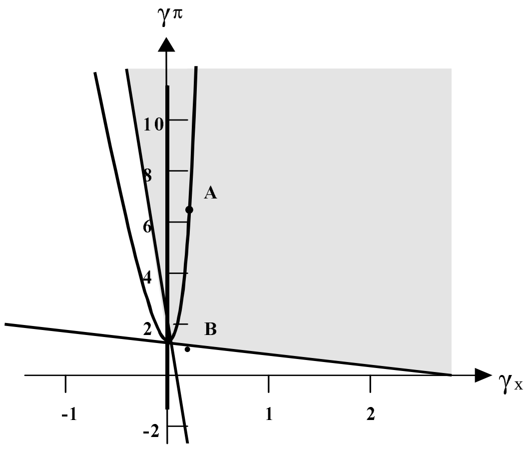
$$\gamma_{\pi}>0.93-41.67\gamma_{x}$$

Graphically, we need to be **inside** the shadowed area.



In the following pictures, the shadowed areas will represent to the combinations of γ_π and γ_x for which we have convergence of least squares learning to rational expectation equilibrium, under the alternative values of the parameters given by Clarida, Gali and Gertler (1999) and by Woodford (1999).





Notice that the "optimal" combinations of γ_π,γ_x with the CGG (2000) parameters, are:
For $\lambda = 0$

$$\begin{aligned}\gamma_\pi^\star &= 4\\ \gamma_x^\star &= 1\end{aligned}$$

For $\lambda = 1$

$$\begin{aligned}\gamma_\pi^\star &= 1.25\\ \gamma_x^\star &= 1\end{aligned}$$

And with the Woodford parameters, the "optimal" combinations of γ_π,γ_x are:
For $\lambda = 0$

$$\begin{aligned}\gamma_\pi^\star &= 7.47\\ \gamma_x^\star &= 0.157\end{aligned}$$

For $\lambda = 1$

$$\begin{aligned}\gamma_\pi^\star &= 1.004\\ \gamma_x^\star &= 0.157\end{aligned}$$

If we look at the graphics we see that for the optimal values we are inside the Determinancy Region in both cases.

6.2 Root-t Convergence

Let consider again the Mapping from PLM to ACL under least square learning hypothesis:

$$T_a\left(A_t'\right) = Q + RA_t$$

and

$$T_b\left(B_t'\right) = \rho_u RB_t + S$$

From Marcet and Sargent (1992) we know that in order to have "root t convergence" we need the eigenvalues of both R and $R\rho_u$, to be less than $\frac{1}{2}$.

Let distinguish again between the two cases:

1. The "Real" Case.

   In this case we need two conditions to be satisfied in order to have convergence to Rational Expectations under Least Squares Learning:

   (a) For Reality

   $$\left(\alpha\varphi\left(1-\gamma_\pi\right)+\beta+\left(1-\varphi\gamma_x\right)\right)^2 - 4\beta\left(1-\varphi\gamma_x\right) > 0$$

   (b) $\boxed{z_1 < 0.5}$
   That is

   $$\frac{\left(\alpha\varphi\left(1-\gamma_\pi\right)+\beta+\left(1-\varphi\gamma_x\right)\right)}{2} + \frac{\sqrt{\left(\alpha\varphi\left(1-\gamma_\pi\right)+\beta+\left(1-\varphi\gamma_x\right)\right)^2 - 4\beta\left(1-\varphi\gamma_x\right)}}{2} < \frac{1}{2}$$

   Let consider what happen under equality:

   $$\gamma_\pi = 1 + \frac{1-2\beta}{2\alpha\varphi} - \frac{1-2\beta}{\alpha}\gamma_x$$

   Notice, that if $z_1$is smaller than $\frac{1}{2}$ than even z_2 is smaller than $\frac{1}{2}$.

   If we assume the values of the parameters of Clarida, Gali and Gertler (1999), the relation

   $$1.24 - 1.32\gamma_\pi + 0.09\gamma_\pi^2 - 0.8\gamma_x + \gamma_x^2 + 0.6\gamma_\pi\gamma_x > 0$$

   shows the combinations of γ_π, γ_x for which the eigenvalues z_1 and z_2 are real.

   In order to have $\boxed{z_1 < \frac{1}{2}}$ we need

   $$\gamma_\pi > -0.33 + 2.67\gamma_x$$

2. The "Complex Case".

   In this case we need two conditions to be satisfied in order to have convergence to Rational Expectations under Least Squares Learning:

   (a) For the solution to be imaginary, we need

   $$(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)<0$$

   (b) $\boxed{\text{Real part of } z_1<\frac{1}{2}}$

   $$\frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2}<\frac{1}{2}$$

   That is

   $$\gamma_\pi>1+\frac{\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}$$

   Notice, that if $z_1$is smaller than $\frac{1}{2}$ than even z_2 is smaller than $\frac{1}{2}$.

   If we assume the values of the parameters of Clarida, Gali and Gertler (1999), the relation

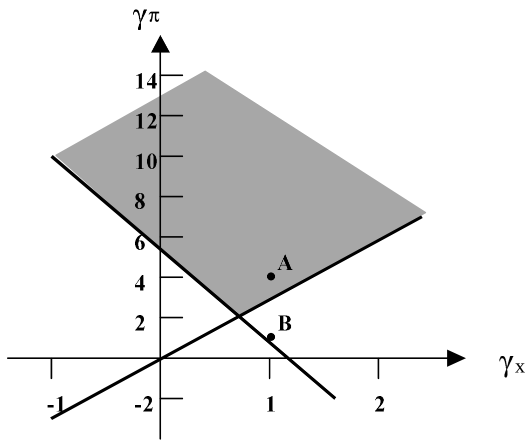
   $$1.24-1.32\gamma_\pi+0.09\gamma_\pi^2-0.8\gamma_x+\gamma_x^2+0.6\gamma_\pi\gamma_x<0$$

   shows the combinations of γ_π, γ_x for which the eigenvalues z_1 and z_2 have an imaginary part.

   In order to have $\boxed{\text{Real part of } z_1<\frac{1}{2}}$ we need

   $$\gamma_\pi>4.0-3.33\gamma_x$$

   Graphically, in order to have root-t convergence we need to be inside the shadowed region.



If we assume the values of the parameters from Woodford (1999), the relation

$$(1.5\,(1-\gamma_\pi)+0.99+(1-(6.37)\,\gamma_x))^2-3.96\,(1-(6.37)\,\gamma_x)>0$$

shows the combinations of γ_π, γ_x for which the eigenvalues z_1 and z_2 are real.

In order to have $\boxed{z_1 < \frac{1}{2}}$ we need

$$\gamma_\pi > -2.205 + 40.83\gamma_x$$

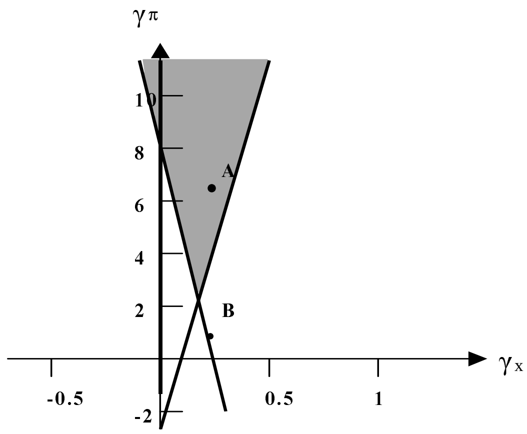
The relation

$$(1.5\,(1-\gamma_\pi)+0.99+(1-(6.37)\,\gamma_x))^2-3.96\,(1-(6.37)\,\gamma_x)<0$$

shows the combinations of γ_π, γ_x for which the eigenvalues z_1 and z_2 are complex.

In order to have $\boxed{\text{Real part of } z_1 < \frac{1}{2}}$ we need

$$\gamma_\pi > 7.47 - 41.67\gamma_x$$



6.3 The mapping from PLM to the ALM

Let's rewrite the conditions for E-stability that we showed in Appendix 1 with the new notation:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = Q' + R' \begin{bmatrix} \widehat{E}_t \pi_{t+1} \\ \widehat{E}_t x_{t+1} \end{bmatrix} + S' u_t$$

$$\left[\begin{array}{c}\widehat{E}_t\pi_{t+1}\\ \widehat{E}_t x_{t+1}\end{array}\right]=A_t'+\rho_u B_t' u_t$$

where

$$A_t'=\left[\begin{array}{c}a_{\pi,t}\\ a_{x,t}\end{array}\right]$$

$$B_t'=\left[\begin{array}{c}b_{\pi,t}\\ b_{x,t}\end{array}\right]$$

$$Q'=\left[\begin{array}{c}\Phi^*\\ \frac{\Phi^*}{\alpha}\end{array}\right]$$

$$R'=\left[\begin{array}{cc}\Gamma^* & 0\\ -\frac{(\beta-\Gamma^*)}{\alpha} & 0\end{array}\right]$$

$$S'=\left[\begin{array}{c}\Psi^*\\ -\frac{(1-\Psi^*)}{\alpha}\end{array}\right]$$

The ALM of inflation and output gap is

$$\left[\begin{array}{c}\pi_t\\ x_t\end{array}\right]=Q'+R'A_t'+\left(\rho_u R' B_t'+S'\right)u_t$$

Thus the Mapping from PLM to ACL takes the form

$$T_a\left(A''\right)=Q'+R'A'$$

and

$$T_b\left(B''\right)=\rho_u R' B'+S'$$

that we could write as

$$T(T_a,T_b)=(Q'+R'A',\rho_u R' B'+S')$$

Expectation Stability is determined by the following matrix differential equation

$$\frac{d}{d\tau}\left(A'',B''\right)=T(A'',B'')-(A'',B'')$$

For this framework E-stability conditions are readily obtained by computing the derivative of $T(A'',B'')-(A'',B'')$ and imposing that the determinant of the matrix with the derivatives of the previous differential equation with respect to A and B is bigger than zero.

$$\det \begin{bmatrix} \Gamma^* - 1 & 0 & 0 & 0 \\ 0 & -\frac{(\beta-\Gamma^*)}{\alpha} - 1 & 0 & 0 \\ 0 & 0 & \rho_u \Gamma^* - 1 & 0 \\ 0 & 0 & 0 & \rho_u - \frac{(\beta-\Gamma^*)}{\alpha} - 1 \end{bmatrix} > 0$$

Since the matrix is diagonal, in this case the condition for E-stability is simply given by:

$$\begin{aligned} \Gamma^* &< 1 \\ \frac{(\Gamma^* - \beta)}{\alpha} &< 1 \\ \Gamma^* &< \frac{1}{\rho_u} \\ \frac{(\Gamma^* - \beta)}{\alpha} &< \frac{1}{\rho_u} \end{aligned}$$

6.4 A comparison of the transition under two different policies

We analize here the derivations of the results showed in section 4

6.4.1 Result 7 - The response of inflation to a cost-push shock

We consider the response of inflation to a positive cost-push shock under the two different policies. Let's assume that at time t,

$$b_{\pi,t}^{ALS} = b_{\pi,t}^{EH} = b_{\pi,t}$$

that is the estimate of b_π is the same under ALS and EH policies. Let's assume that up to time t $u_t = 0$. In t we have $\varepsilon_t > 0$ that implies $u_t = \varepsilon_t > 0$. Let's assume that from $t+1$ on, $\varepsilon_t = 0$. Let's look at the derivative of the inflation with respect to the shock, conditional to the estimation of the learning coefficients at time t being the same under the two policies:

$$\frac{\partial \pi_t^{EH}}{\partial u_t} = \rho_u \Gamma^* b_{\pi,t} + \Psi^*$$

$$\frac{\partial \pi_t^{ALS}}{\partial u_t} = \rho_u \Gamma' b_{\pi,t} + \frac{\Psi \left(1 - \rho_u \Gamma'\right)}{\left(1 - \rho_u \Gamma^*\right)}$$

and for

$$\Gamma' < \Gamma^*$$

$$\Delta\pi_t^{ALS} > \Delta\pi_t^{EH} \quad \text{IF} \quad b_{\pi,t} < b_\pi$$

while

$$\Delta\pi_t^{ALS} < \Delta\pi_t^{EH} \quad \text{IF} \quad b_{\pi,t} > b_\pi$$

6.4.2 Result 8 - The response of output-gap to a cost-push shock

We consider the response of inflation to a positive cost-push shock under the two different policies. Let's assume that at time t,

$$b_{\pi,t}^{ALS} = b_{\pi,t}^{EH} = b_{\pi,t}$$

that is the estimate of b_π is the same under ALS and EH policies. Let's assume that up to time t $u_t = 0$. In t we have $\varepsilon_t > 0$ that implies $u_t = \varepsilon_t > 0$. Let's assume that from $t+1$ on, $\varepsilon_t = 0$. We consider the derivative of the output gap with respect to the shock, conditional to the estimation of the learning coefficients at time t being the same under the two policies::

$$\frac{\partial x_t^{EH}}{\partial u_t} = -\rho_u \frac{(\beta - \Gamma^*)}{\alpha} b_{\pi,t} - \frac{(1 - \Psi^*)}{\alpha}$$

$$\frac{\partial x_t^{ALS}}{\partial u_t} = -\frac{(\beta - \Gamma')}{\alpha} b_{\pi,t}\rho_u - \frac{\left(1 - \Psi^* \left(1 - \rho_u \Gamma^*\right)^{-1} \left(1 - \rho_u \Gamma'\right)\right)}{\alpha}$$

and for

$$\Gamma' < \Gamma^*$$

$$\Delta x_t^{ALS} > \Delta x_t^{EH} \quad \text{IF} \quad b_{\pi,t} < b_\pi$$

while

$$\Delta x_t^{ALS} < \Delta x_t^{EH} \quad \text{IF} \quad b_{\pi,t} > b_\pi$$